Statistics of Poincaré recurrences for a class of smooth circle maps

Nikola Buric [1], Aldo Rampioni [2] and Giorgio Turchetti [2]

(1) Department of Physics and Mathematics,
Faculty of Pharmacy, University of Beograd,
Vojvode Stepe 450,
11000 Beograd, Yugoslavia.
(2) Department of Physics, University of Bologna,
INFN Sezione di Bologna, Italy.

February 3, 2008

Abstract

Statistics of Poincaré recurrence for a class of circle maps, including sub-critical, critical, and super-critical cases, are studied. It is shown how the topological differences in the various types of the dynamics are manifested in the statistics of the return times.

PACS:

1   Introduction

Statistics of the Poincaré recurrences, i.e. return times statistics, has recently gained renewed importance in the theory of dynamical systems, primarily due to the fact that it could be used as an indicator of the statistical properties of the system's global dynamics on large parts of its phase space (see for example [1],[2]). For instance, the first return time can be used to calculate the metric entropy of a system with an ergodic invariant measure [3], and seems to be related to other generalized dimensions used to describe the fractal properties of the dynamics, at least for certain types of dynamical systems [4]. In this paper we study the Poincaré recurrences for maps of the circle, that can display quasi-periodic, bi-stable or chaotic dynamics, depending on the values of their two parameters.

Given a discrete dynamical system on a phase space M with a transformation $T : M \to M$ and a reference measure μ on M, the first return time, in a measurable set $A \subset M$, of a point $x \in A$, is defined by

$$\tau_A(x) = \inf_k \{T^k(x) \in A\}. \tag{1}$$

The first return time τ_A for the set A, and the average return time $< \tau_A >$ for the set A, are the following

$$\tau_A = \inf_{x \in A} \tau_A(x), \qquad < \tau_A >= \int_A \tau_A(x) d\mu_A(x), \tag{2}$$

where $\mu_A(x)$ is the conditional measure $\mu_A(B) = \mu(B)/\mu(A)$ for any $B \subseteq A$.

With $< \tau_A >$ and $\tau_A(x)$ we define the probability $F_A(t)$ that the normalized first return time in the set A is larger than t for the points in the set A

$$F_A(t) = \mu_A(A_{>t}) \qquad A_{>t} \equiv \{x \in A_t : \ \tau_A(x)/\langle \tau_A \rangle > t\} . \tag{3}$$

A limit probability measure $F_x(t)$ may be associated to any point x by considering a sequence of neighborhoods A_k of x with $\mu(A_k) \to 0$ as $k \to \infty$. For ergodic systems the limit is the the same for almost every point and in

this case the average return time in a domain A is given by

$$\langle \tau_A \rangle = \frac{1}{\mu(A)} \tag{4}$$

according to the well known Kac's lemma. For some classes of hyperbolic dynamical systems [5] it has been proved that the return times spectrum follows the exponential-one decay law $F_x(t) = e^{-t}$ at almost every x. If x belongs to the dense set of the unstable periodic points then $F_x(t) = a \exp(-\alpha t)$ where a and α depend on the period. It is known that the properties of the distribution of the return times can give criteria for the existence of an equilibrium and the rates of mixing [7]. A polynomial decay law for $F_x(t)$ was found for integrable area-preserving maps [8][9]. On the basis of numerical computations, the polynomial decay of return times spectra $F_x(t)$ was also claimed to be a generic property of Hamiltonian systems with mixed phase space, where complicated self similar fractal structures are present [10],[1]. In fact, for such systems, a convex combination of the power law and the exponential law decays seems to provide a very good fit of $F_x(t)$ and is theoretically justified [11].

Our aim is to study the statistics of the first return times for smooth perturbations (non necessarily invertible) of the uniform rotations $T_{k,\Omega}$ on the circle S^1,

$$\theta \rightarrow T_{k,\Omega}(\theta) = \theta + \Omega + \frac{k}{2\pi} f(\theta), \tag{5}$$

where $\theta \in S^1$ and $k \in R^+$ and $\Omega \in R^+$ are the parameters of the map. The function $f(\theta)$ is a trigonometric polynomial such that the maps are monotonic for $k \in [0, 1)$ and non-invertible for $k > 1$. The details of the dynamics and the structure of the bifurcation diagram have been thoroughly studied for the sine-circle map given by $T_{k,\Omega}(\theta) = \theta + \Omega + \frac{k}{2\pi} \sin 2\pi\theta$ (see for example [14]- [26]). Other families of the form (4) are used to study the universality of the properties found for the sine-circle map.

The main results of our analysis of the recurrence times for the circle maps of the form (5) can be summarized as follows. For $k < 1$, where the map

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is invertible and diffeomorphic to a rotation, if the rotation number is diophantine, three return times are observed. This is in agreement with Slater's theorem [12], which we extend from linear rotations to the diffeomorphisms of the circle, see section 2. At the critical value $k = 1$ three return times are still observed and the average return times allow an effective reconstruction of the invariant measure, see section 3. In the super-critical case (section 4), we show the appearance of a continuous spectrum $F_\theta(t)$ for maps that have chaotic orbits at $k = 1 + \epsilon$ (section 4.1), and for maps $T_{k,\Omega}$ for k beyond the corresponding accumulation point of period-doubling bifurcations (section 4.3). Also, the properties of the return times could be used to detect the existence of attracting periodic orbits (section 4.2).

Our conclusion is that spectrum reflects, the bifurcations in the topological properties of the dynamics. and is a useful tool to investigate them.

2   Dynamical properties of the circle map

Let us briefly recapitulate some of the properties of the circle maps (5). For our purposes, we distinguish three regions depending on the parameter k: the sub-critical region $k < 1$, the weakly super-critical region $k = 1 + \epsilon$ and the strongly super-critical region $k >> 1$.

2.1   Sub-critical and critical region

For $k \leq 1$ the circle map $T_{k,\Omega}$ is an orientation preserving homeomorphism of the circle, and for $k < 1$ the map is a diffeomorphism. In any case, its topological properties are fixed by the rotation number ω defined by

$$\omega = \lim_{n \to \infty} \frac{\bar{T}^n(\theta)}{n}, \tag{6}$$

where $\bar{T}$ is the lift of T on the real line, and $0 \leq \omega < 1$ for the definition to be unique.

The map $T_{k,\Omega}$, for $k < 1$ is conjugate to the linear rotation by the angle $\omega(k, \Omega)$. The properties of the conjugation $\Theta = \Phi_{k,\Omega}(\theta)$ with the linear rotation $R_\omega : \Theta \to \Theta + \omega$ depend on the arithmetic properties of ω. For a generic rotation number $\Phi_{k,\Omega}$ is a homeomorphism according to Denjoy's theorem [14].

Furthermore if ω is a Diophantine irrational the conjugation $\Phi_{k,\Omega}$ is a diffeomorphism [15]. In such a case the topological and metric properties of the linear rotation R_ω extend to the map $T_{k,\Omega}$ by using

$$T_{k,\Omega}^n = \Phi_{k,\Omega}^{-1} \circ R_\omega^n \circ \Phi_{k,\Omega} \tag{7}$$

Slater's theorem [12], stating that for any irrational linear rotation, and any connected interval there are at most three different return times, one of them being the sum of the others, extends from linear rotations to the map $T_{k,\Omega}$. Two of the three return times are always the consecutive denominators in the continued fraction expansion of the irrational rotation number ω, and one of the return times is a sum of the other two. Two points θ, $T_{k,\Omega}^n(\theta)$ in a connected interval A are mapped into $\Theta = \Phi_{k,\Omega}(\theta)$ and $R_\omega^n(\Theta)$ of the connected interval $\Phi_{k,\Omega}(A)$

A critical map $T_{k=1,\Omega}$ is only a homemorphism of the circle but is still characterized by a unique rotation number. Our calculations indicate that, at least, for sufficiently irrational rotation numbers there are still only three return times, like in the sub-critical case.

2.2   Weakly super-critical region

When $k > 1$ the map ceases to be invertible and it does not have a unique rotation number. For a given $(k > 1, \Omega)$ quasi-periodic, chaotic, at most two stable periodic orbits, and orbits asymptotic to the latter can coexist. The points in the parameters plane (k, Ω) that correspond to sub-critical maps with rational P/Q rotation numbers form domains, the P/Q tongues, that can be extended above the critical line $k = 1$ where they start to overlap.

The boundaries of the tongues, correspond to tangent bifurcation, and can be found by solving for θ and Ω the equations

$$T_{k,\Omega}^{(Q)}(\theta) = \theta + P, \qquad T_{k,\Omega}^{(Q)}(\theta)'(\theta) = 1, \qquad (8)$$

where $T_{k,\Omega}^{(Q)}(\theta)$ is the Q-th application of the map, $T_{k,\Omega}^{(Q)}(\theta)'$ its derivative (see for example [19]).

The union of all P/Q tongues at $k = 1$ has full measure, and for $k = 1+\epsilon$ any map (k, Ω) belongs to intersection of two tongues P_1/Q_1 and P_2/Q_2 that correspond to Farey neighbours with sufficiently high Q_1 and Q_2. Actually, (k, Ω) is in the intersection of the tongues that correspond to all rationals deeper in the Farey tree and between P_1/Q_1 and P_2/Q_2. To predict the dynamics of the map one has to know a very complicated fine structure of bifurcations inside the overlapping tongues. The lines in the $(k < 1, \Omega)$ plane that correspond to maps with irrational rotation numbers become, for $k > 1$, domains such that a map in such a domain have at least one orbit with the corresponding irrational rotation number, but also has orbits with other rotation numbers consistent with the P/Q tongues that overlap, and chaotic orbits [20]. Weakly super-critical maps of the form (5) with chaotic orbits have been used to study the quasi-periodic rout to chaotic dynamics [18], [21],[22].

2.3 Strongly super-critical region

Besides the maps (k, Ω) with the small $k > 1$ that have chaotic orbits, there are such Ω that maps (k, Ω) inside a P/Q tongue will have strongly chaotic behaviour only if $k > K_c(P/Q)$, where $K_c(P/Q)$ is the value of k at which the period doubling bifurcations inside P/Q tongue accumulate, which could be quite large. For example, for $\Omega = 0$ the critical value of k is estimated to be $k = K_c(0) = 4.60366$ [16]. We shall see that the properties of return times change abruptly at $k = K_c(P/Q)$, and can be used to estimate $K_c(P/Q)$.

3 Return times for sub-critical and critical dynamics

The numerically observed return times for $k < 1$ and diophantine rotation numbers ω are in agreement with a straightforward extension of Slater's theorem. The same result is found at the critical value $k = 1$. For generic interval and each quadratic irrational ω that we have studied, there are again only three return times, one is the sum of the other two and two always coincide with the denominators of the corresponding two successive approximants of ω, see for example figure 1a. We are fairly confident that the conclusions are valid for any quadratic ω and the maps in the class (5), but we can not claim anything for homeomorphisms with a generic rotation number (see the reference [28]).

The three return times, and their relative weights depend on the location and the size of the interval. However, there is a sequence of intervals, obtained by partitioning the circle with the iterates of an initial point θ_0 that is best suited for the analysis of the return times at a point θ_0 of the map with a given irrational rotation number ω. One considers the trajectory formed by the $q_{i-1} + q_i - 1$ successive iterates of a point θ_0, where p_i/q_i are rational continued fraction approximates of the ω.

The q_i-th and the q_{i-1}-th iterate form the boundary of an interval $A_i(\theta_0)$ which contains only the initial point θ_0 and no other points of the considered part of the trajectory. The points generated by $q_{i+1} + q_i - 1$ iterates will subdivide the intervals generated by $q_i + q_{i-1} - 1$ iterates. One obtains a sequence $A_{i+1} \subseteq A_i$ of intervals that converge to the point θ_0 on the orbit of $T_{k,\Omega}$. Calculating the return time τ_{A_i} for such sequence of intervals A_i, $i \to \infty$ is best adopted to the calculation of the return time at the point $\theta_0 \in \bigcap_{i=1}^{\infty} A_i$. Suppose that the q_{i-1}-th iterate is to the left of θ_0 which is to the left of the q_i-th iterate. Then, each of the points $\theta \in A_i(\theta_0)$ that are on the left side of θ_0 have a unique first return time equal to q_i, and points $\theta \in A_i(\theta_0)$ to the

right of θ_0 also have a unique first return time equal to q_{i-1}. Thus, there are only two return times equal to q_{i-1} and q_i. Furthermore, the union of the intervals formed by $q_{i+1} + q_i - 1$ iterates gives a partition $\mathcal{P}_i$ of the circle, and the return times into various intervals of a partition $\mathcal{P}_i$ are the same q_i and q_{i-1}, although the relative weights are obviously different for a nonlinear map. However, as $\imath \to \infty$, and for an irrational number $\omega = [a, a, a, \ldots]$ with a constant tail of the continued fraction expansion, the relative weights of the two return times become independent of i, which implies the existence of a point spectrum independent of θ (see Appendix)

$$F(t) = w_- \delta(t - t_-) + w_+ \delta(t - t_+), \tag{9}$$

where

$$t_- = \lim_{\imath \to \infty} \frac{q_i}{< \tau_{A_i} >} = \frac{1 + a + \omega}{1 + 2\omega}, \qquad t_+ = \lim_{\imath \to \infty} \frac{q_{i-1}}{< \tau_{A_i} >} = \omega t_-, \tag{10}$$

are the renormalized return times in the limit and q_i are denominators of the continued fraction approximants of ω. Since the map is smoothly conjugated for $k < 1$ and diophantine ω to a linear rotation, the above properties follow from a proposition we prove in this case in the Appendix.

Approximation of the measure: The distribution of the return times in various intervals can be illustrated using the mean return time. In fact, for any homeomorphism of the circle there is a unique invariant ergodic measure which is absolutely continuous with respect to the Lebesgue measure on the circle, and the density of this measure can be obtained using return times, as is indicated by the Kac lemma. In figures 1b,c,d we plot density of a coarse-grained mean return time, i.e. the ratio between uniform average of the return times into an interval of a partition of the circle divided by the Lebesgue measure of the interval, for a sub-critical and the critical sine-circle map. For a sufficiently fine partition this quantity illustrates the density of the unique invariant ergodic measure for the considered map. The main computational cost of this method is due to the computational time,

complementary to the perturbative method where the main requirement is for a sufficient storage space. In order to achieve an resolution of 10^{-m} by the return times calculations one needs roughly $10^m \times K10^m \times 10^{2m}$ iterations of the map. Here, 10^m is the number of intervals of the partition, $K10^m$ with $K < 10$ is roughly the average return time and 10^{2m} is the number of points in each of the intervals. The same space resolution is archived by using 10^m Fourier components of the conjugation function in the perturbation method. For 1 percent accuracy, i.e. $m = 2$, both methods can be easily implemented, but 10^{-3} accuracy represents a more challenging task for both the methods.

For $k < 1$ the density of the invariant measure is a smooth function (see figure 1b,c), and for the critical maps the density becomes singular (figure 1d) [25],[26]. Fractal properties of the ergodic measures for the critical circle maps (5) have been studied. There is strong evidence [26] that the class of critical maps with the same fractal spectrum of the invariant measure are characterized only by the rotation number (actually probably only by the tail in it's continued fraction expansion [27]) and by the type of the singularity that induces the critical behaviour.

4 Return times in the super-critical cases

A super-critical map could have chaotic orbits, at most two stable attracting orbits and orbits asymptotic to these. Our numerical computations support a conclusion that for any Ω there is sufficiently large k, such that the distribution of the first return times is given by exponentially fast decay. Furthermore, there are maps $T_{k,\Omega}$ which have chaotic orbits for $k = 1 + \epsilon$ with arbitrary small but non-zero ϵ, and maps $T_{k,\Omega}$ that show chaotic behaviour only for sufficiently large $k > K_c(P/Q)$. This is manifested by two different roots to the exponential decay of the distribution of the first return times.

The following three typical situations can be clearly distinguished by studying the properties of the return times.

4.1 Quasi-periodic route

Consider, first, a weakly-super-critical map (k, Ω) with $k = 1 + \epsilon$, where ϵ is arbitrary small but non-zero, and with Ω such that there is an orbit with an irrational rotation number. For example, suppose that the rotation number $\omega(\theta_0)$ of the orbit through point $\theta_0 = 0$ is numerically equal to the golden mean γ. We can use the return times into a sequence of nested intervals that shrinks to θ_0, or analogous sequences that shrink on other points to study the dynamics.

Results of such analysis are shown in figures 2a,b,c,d which illustrate the dynamics of the same map (k_0, Ω_0) but as revealed by the statistics of the return times into a sequence of nested intervals of decreasing size. Here $k_0 = 1.01$ and $\Omega_0 = 0.606494989$, leading to an orbit with ω whose continued fraction expansion is $\omega = [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, \ldots]$, and which has denominators of the convergents $q_i = 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \ldots$.

The fact that the map is not topologically equivalent to a uniform rotation is manifested already at the resolution given by the interval of finite size. For example, for the interval $(0, 0.0015)$ (fig. 2a) there are four return times $55, 89, 144, 199$, where the fourth 199 is not a denominator of any of the convergents to ω. However, it is the denominator of the Farey neighbor $123/199$ of the approximant $89/144$. This signals that, for $k = k_0$ the $89/144$-tongue and the $123/199$-tongue have common part of their interiors, and that the point (k_0, Ω_0) belongs to both tongues. The interval that will detect the existence of two intersecting tongues at (k_0, Ω_0) must be smaller than the distance between θ_0 and its 144-th iterate. The first return times into a larger interval are at most 144 for all its points, i.e. with the resolution weaker than the points of the first 144 iterates of θ_0 the map looks as a smooth rotation.

Further analysis of the return times for the same map (k_0, Ω_0) but into smaller intervals reveals intersections of tongues at the deeper levels of the Farey tree between $89/144$ and $123/199$. This is illustrated in figures 2b,c

which show the return times $89, 144, 199, 233, 288, 343$ into the interval $(0, 0.001)$, and a very large number of return times into the interval $(0, 0.0001)$. The statistics of the return times into $A = (0, 0.0001)$ is illustrated in figure 2d, by plotting the logarithm of the probability density $\ln F_{\theta_0}(t)$ of the return time larger that $t = \tau/ < \tau_A >$ versus the normalized time t. The distribution is given by exponential decay with an exponent that is numerically close to 1. Thus, on the sufficiently small scale, the map has the distribution of the return times characteristic of strongly chaotic systems.

Increasing k, and changing Ω so that there is always an orbit with rotation number γ, moves the point (k, Ω) into the domain were the intersections of tongues, and the chaotic behaviour, can be detected using larger intervals. Figures 3a,b,c, show the effects of more intersected tongues on the distribution of return times into the same interval $A = (0, 0.0015)$. For $k = 1.2$, the return times and their relative weights are such that the distribution $F_{\theta_0}(t)$ is given by exponential decay with the exponent that is again numerically equal to 1. In fact, for any $k > 1$ one can find pairs (k, Ω) which imply non-periodic orbit through the point θ_0, and the the density $F_{\theta_0}(t)$ for such map (k, Ω) is $F_{\theta_0}(t) = \exp(-t)$. This is illustrated in figure 4a for $k = 1.3; 3; 4; 5$ and $k = 6.14$ and the corresponding Ω. For all these maps and for all tested θ the distribution $F_{\theta}(t)$ is always $F_{\theta_0}(t) = \exp(-t)$.

Numerical evidence, presented in figures 2 and 3, suggests the following conjecture: Suppose that a point θ_0 lies on a quasi-periodic or on a chaotic orbit of a map $(k = 1 + \epsilon, \Omega)$, for arbitrary $\epsilon \neq 0$ and consider a sequence A_i of nested intervals containing θ whose length approaches zero by increasing i. The number of different return times also increases with i, and asymptotically, as A_i shrink to θ, the distribution of return times becomes continuous. Furthermore, the density of probability $F_{\theta}(t)$ with respect to the Lebesgue measure on the circle of the normalized first return time larger than t is given by exponential decay.

4.2 (Bi)-stability

Possible existence of attracting periodic orbits can be detected by studying the return times into various intervals of a single partition of the circle. Although there could be no bounded invariant density in this case the return times are still well defined. The return times into different intervals depend on whether the stable orbit have points in the considered interval or not. If there is an attracting periodic orbit with no points in the interval than, obviously, there are points in the interval that will never come back, i.e. with the first return time $\tau = \infty$, indicating the existence of the attracting orbit. In this case there will be only those return times that correspond to orbits that re-visit the interval at least once before being attracted to the attracting periodic orbit. On the other hand, for examples of chaotic maps, illustrated in the figure 4a, the return time statistics for increasingly fine partitions confirms that these maps have no attracting periodic orbits.

4.3 Period-doubling route

In order to study how the period-doubling route to an ultimately chaotic map $(k, \Omega = P/Q)$ is manifested in the properties of the return times, consider the maps $(k, \Omega = 0)$ for various k. Results are illustrated by various curves in figure 4. The return times into intervals at $\theta_0 = 0$ for any $k \in (1, K_c(\Omega = 0)$ show the existence of stable periodic orbits, as described in the previous subsection. Suddenly, at the accumulation points of the period-doubling cascade $k = K_c(\Omega = 0)$, the distribution of the return times becomes continuous. For such critical k, the distribution $F_{\theta_0}(t)$ at the point $\theta_0 = 0$ is given by the exponential decay $a \exp(-\alpha t)$, with $a < 1$ and $\alpha \approx 0.8 \neq 1$. Furthermore, still for $k = K_c(\Omega = 0)$ the return times into intervals at θ_0 of finite size are given by double exponential curves, which converge to the single exponential $\exp(-\alpha t)$ as the intervals shrink to θ_0. For example, for the interval

$A = (0, 0.01)$ the distribution $F_A(t)$ is well approximated by

$$F_A(t) = 0.15 \exp(-0.257t) + 0.7 \exp(-1.8t) \tag{11}$$

represented by the dash-ed line in figure 4b.

Other curves in figure 4 represent the distributions of the return times for examples of strongly super-critical maps, and for intervals of decreasing size around different points. The curves with the unique slope in figure 4b are the distributions for maps in the tongues with $\Omega = 0$ for k beyond the accumulation of period doublings, i.e. for $k > 4.604$ and with $\Omega = 1/2$ for $k > 1.978$. In all the cases, the distribution is given by exponential decay that converges to $F_{\theta_0}(t) = a \exp(-\alpha t)$, where $\alpha \neq 1$ and θ_0 is on an unstable periodic orbit. Non-linear curves in figure 4b represent the distributions for the map at the accumulation of period doublings and on the indicated intervals of decreasing size. The curves in 4a represent $F_\theta(t)$ for examples of strongly super-critical maps at θ not on a periodic point, when $F_{\theta_0}(t) = \exp(-\alpha t)$ with $\alpha \approx 1.0$. Shown are examples with $k = 1.3, 3, 4, 5, 6.14$ and the corresponding Ω as explained earlier, and also examples of maps with $\Omega = 0; 1/2$ and large k. All this is consistent with the statistics of the return times for other examples of strongly chaotic maps.

5   Summary and conclusions

We have analyzed the circle map numerically for a wide range of values in the parameter space (k, Ω) and different sets of initial conditions. The results can be summarized as follows.

Sub-critical and critical region: For $k < 1$ three return times are observed and this is theoretically and numerically justified. For diophantine rotation numbers ω this result can be presented as a corollary of Slater's theorem, since the map is diffeomorphic to a linear rotation, to which such a theorem applies. For the case of a special sequence of nested intervals including a given point we provide a very simple proof of Slater's theorem, showing

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that there are only two return times and proving the existence of a limit point spectrum $F(t)$. The critical dynamics $k = 1$ is also clear since three return times for a generic interval and for each quadratic irrational rotation number are observed. For $k \leq 1$ and ω diophantine a piece-wise constant approximation to the invariant measure is obtained from the average return times from a uniform partition of the circle with a very simple procedure and an accuracy comparable to other methods.

Super-critical region: The dynamics of a map (k, Ω) in the weak super-critical case $k = 1 + \epsilon$ is dictated by the P/Q tongues with a nonempty intersection that contains the point (k, Ω). This is manifested, and could be detected, in the distribution of the first return times by appearance of more than three return times, which correspond to the rationals on the Farey tree in-between the the two major overlapping tongues that contain (k, Ω). In the case that the interval contains a point on a non-periodic orbit than there is a sub-interval such that the distribution, with respect to the uniform distribution of initial points, of the return times into this sub-interval is typical for strongly chaotic systems, i.e. the exponential decay with exponent equal to unity.

In the intermediate region two return times or the exponential-one spectrum are observed depending on the existence of attracting periodic orbits. The way $F_A(t) \to e^{-t}$ when the size of A containing θ_0 approaches 0 depends on (k, Ω, θ_0) and it is convenient to distinguish various routes.

Quasi -periodic route: For any non-periodic point θ_0 of the map (k, Ω) with $k \geq 1 + \epsilon$ where $\epsilon > 0$ is arbitrary small, the spectrum is exponential-one. Choosing a finite interval A the spectrum $F_A(t)$ appears as continuous for a sufficiently small A.

Bi-stability: For $k = 1 + \epsilon$ there are values of (k, Ω) such that the map has attracting periodic orbits. For any interval A not intersecting one of these orbits one of the return times is $\tau = \infty$, since many points do not return being attracted by the periodic orbit. For $1 < k < 2$ the attractive periodic

orbits are present for most values of Ω.

Period-doubling route: For any rational $\Omega = p/q$ there is a critical $k_c = k_c(p/q)$ corresponding to the accumulation of period-doublings in every tongue. The transition from (bi)-stability to chaoticity is manifested abruptly in the spectrum $F_A(t)$. For $k = k_c$ the spectrum $F_A(t)$ is continuous and there are intervals such that it can be fitted with a double exponential: in the limit $\mu(A) \to 0$ the spectrum becomes exponential.

Strongly super-critical region For $k \gg 1$ the map is chaotic and the periodic orbits are unstable. The spectrum $F(t)$ is exponential-one except for a set of points of measure zero corresponding to unstable periodic orbits where the exponential decay speed is different from 1. The results for the super-critical dynamics indicate that the analysis of the return times spectra in the super-critical case could be a useful tool for a better understanding of the transition from the weakly to the strongly chaotic regime. It would be interesting to follow in details the pattern of intersections of tongues and period doubling bifurcations inside each tongue, leading to the strongly super-critical case, by using the return times spectra and the way they are approached when a sequence of nested intervals squeezing to a point is considered. To conclude the computation of the return times spectrum is a simple and effective way to explore a dynamical system and its bifurcations.

6   Appendix

Letting R_ω be the linear map conjugated to $T_{k,\Omega}$ and $\Theta_0 = \Phi^{-1}(\theta_0)$ be the image of the initial point θ_0 we consider a partition of the circle $R_\omega^n \Theta_0$ to which corresponds another partition $T_{k,\Omega}^n(\theta_0)$. The order in these partitions

is preserved since the maps are diffeomorphic for $k < 1$ and ω diophantine Let the continued fraction expansion of ω be given by

$$\omega = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots}}} \tag{12}$$

and $p_i/q_i = [a_1, \ldots, a_i]$ be the corresponding rational approximations of order i. The odd and even approximants are upper and lower bounds to ω, converging monotonically to it. The following recurrence relations hold

$$p_i = a_i\, p_{i-1} + p_{i-2} \qquad\qquad q_i = a_i\, q_{i-1} + q_{i-2} \tag{13}$$

and the following inequalities hold

$$\frac{1}{q_{i-1} + q_i} \leq p_{i-1} - \omega\, q_{i-1} \leq \frac{1}{q_i} \qquad \frac{1}{q_i + q_{i+1}} \leq q_i\, \omega - p_i \leq \frac{1}{q_{i+1}} \tag{14}$$

Denoting the linear map iterates of a point Θ_0 by

$$\Theta_0^{q_i} = R_\omega^{q_i}\, \Theta_0 = \Theta_0 + q_i\, \omega \bmod 1 = \Theta_0 + q_i\, \omega - p_i \tag{15}$$

the odd and even sequences $\Theta_0^{q_{2i-1}}$ and $\Theta_0^{q_{2i}}$ converge monotonically from below and from above to Θ_0. The intervals $A_i(\Theta_0) = [\Theta_0^{q_{i-1}}, \Theta_0^{q_i}]$ if i is even and $A_i(\Theta_0) = [\Theta_0^{q_i}, \Theta_0^{q_{i-1}}]$ if i is odd, form a nested sequence of intervals $A_1 \supset \ldots A_i \supset A_{i+1} \ldots$ squeezing to Θ_0 as $i \to \infty$ and from the previous inequalities the following inclusions hold.

$$\left[\Theta_0 - \frac{1}{q_{i-1} + q_i},\ \Theta_0 + \frac{1}{q_i + q_{i+1}}\right] \subset A_i(\theta_0) \subset \left[\Theta_0 - \frac{1}{q_i},\ \Theta_0 + \frac{1}{q_{i+1}}\right] \tag{16}$$

The intervals $B_i = \Phi_{k,\Omega}^{-1}(A_i) = [T_{k,\Omega}^{i-1}(\theta_0), T_{k,\Omega}^i(\theta_0)]$ enjoy the same properties for any diophantine ω since the map is orientation preserving and $\Phi_{k,\Omega}$ is a diffeomorphism. The sequence B_i is a nested monotonic sequence of intervals approaching θ_0 exponentially fast. According to Kac's lemma the average return times for the intervals B_i and A_i are given by the inverse of their length, which increases to 0 exponentially fast with i.

According to Slater's theorem for a generic interval and a linear map with irrational ω, there are three return times, the last one being the sum of the first two. The sequences of nested intervals $[\Theta_0^{q_{i-1}}, \Theta_0]$ and $[\Theta_0, \Theta_0^{q_i}]$ enjoy this property. For the intervals A_i defined above the return times are only two and we give a sketch of the proof since it quite simple.

Proposition: The return times in the interval $A_i(\Theta_0)$ for the linear map with an irrational ω are two. If i is even (odd) and for $\Theta \in A_i(\Theta_0)$ the return time is q_i (q_{i-1}) if $\Theta < \Theta_0$ and q_{i-1} (q_i) if $\Theta > \Theta_0$.

Proof The proofs for even or odd i are analogous so we consider only the even i. In order to prove the results we suppose Θ_0 is sufficiently far from the identified ends 0 and 1 that the order preserving relation becomes the inequality between real numbers. For initial conditions near 0, the torus defined as the interval $[-1/2, 1/2]$ with identified ends should be considered to have the same correspondence. The Lagrange's theorem states that in the interval $1 \leq n \leq q_i + q_{i-1} - 1$ the minimum of $|n\omega - m|$, where $m = [n\omega]$ is the integer part, is reached for $n = q_i$. As a consequence, the two points Θ_0^n closest to Θ_0 for $n \leq q_i + q_{i-1} - 1$ correspond to $n = q_{i-1}$, q_i. We consider the sequence of nested intervals $A_i(\Theta_0) = [\Theta_0^{q_{i-1}}, \Theta_0 q_i]$ and for a fixed i a point $\Theta \in A_i(\Theta_0)$ we denote by $\Theta^n = R_\omega^n \Theta$ the points of its orbits and examine two possible cases.

Case 1   If $\Theta < \Theta_0$ then: $\Theta^n \notin A_i$ for $n < q_i$, $\Theta^{q_i} \in A_i$.

To prove the first point we notice that

$$\Theta_0^{q_i} - \Theta^{q_i} = \Theta_0 - \Theta > 0 \tag{17}$$

and

$$\Theta^{q_i} - \Theta_0^{q_{i-1}} = \Theta^{q_i} - \Theta^{q_{i-1}} - (\Theta_0^{q_{i-1}} - \Theta^{q_{i-1}}) = \mu_L(A_i) - (\Theta_0 - \Theta) > 0 \tag{18}$$

since both Θ_0, Θ belong to A_i. To prove the second property we notice that the points Θ_n closest to Θ for $n < q_i$ are met for $n = q_{i-2}$, q_{i-1} and $\Theta^{q_{i-1}} < \Theta < \Theta^{q_{i-2}}$ and we show that both are out of A_i.

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$$\Theta_0^{q_{i-1}} - \Theta^{q_{i-1}} = \Theta_0 - \Theta > 0 \tag{19}$$

To prove $\Theta^{q_{i-2}} > \Theta_0^{q_i}$ we show that $\Theta^{q_{i-2}} - \Theta \geq \mu_L(A_i)$. Indeed $\mu_L(A_i) = \omega q_i - p_i - \omega q_{i-1} - p_{i-1} = \omega(q_i - q_{i-1}) - (p_i - p_{i-1})$

$$= \omega q_{i-2} - p_{i-2} + (a_i - 1)(\omega\, q_{i-1} - p_{i-1}) \tag{20}$$

Consequently we obtain

$$\Theta^{q_{i-2}} - \Theta = \omega q_{i-2} - p_{i-2} = \mu_L(A_i) + (a_i - 1)(p_{i-1} - \omega\, q_{i-1}) \geq \mu_L(A_i) \tag{21}$$

the equal sign holding for the golden mean.

Case 2   If $\Theta > \Theta_0$ then: $\Theta^n \notin A_i$ for $n < q_{i-1},$ $\Theta^{q_{i-1}} \in A_i.$

The fist point is proved observing that

$$\Theta^{q_{i-1}} - \Theta_0^{q_{i-1}} = \Theta - \Theta_0 > 0 \tag{22}$$

and

$$\Theta_0^{q_i} - \Theta^{q_{i-1}} = \Theta_0^{q_i} - \Theta^{q_i} + \Theta^{q_i} - \Theta^{q_{i-1}} = \mu_L(A_i) - (\Theta - \Theta_0) > 0 \tag{23}$$

Concerning the last point we notice that we already proved (two equations above) that $\Theta^{q_{i-2}} - \Theta > \mu_L(A_i)$ for any $\Theta \in A_i$. We prove also that $\Theta - \Theta^{q_{i-3}} > \mu(A_i)$. Indeed from $\Theta - \Theta^{q_{i-3}} - \mu(A_i) = p_{i-3} - \omega q_{i-3} - (\omega q_i - p_i) + (\omega q_{i-1} - p_{i-1}) =$

$$= p_i - a_{i-1}\, p_{i-2} - \omega\,(q_i - a_{i-1}q_{i-2}) \geq p_i - p_{i-2} - \omega(\, q_i - q_{i-2})$$
$$= a_i(p_{i-2} - \omega\, q_{i-1}) > 0, \text{ where we have used twice the}$$

recurrence relations for the p_i and q_i, taking into account that $\omega\, q_{i-2} - p_{i-2} > 0$, and that $a_i \geq 1$ since we have assumed ω to be irrational.

Existence of the spectrum: The return times in the intervals A_i are given by q_i if $\Theta \in A_i^- \equiv A\Theta \in [\Theta_0^{q_{i-1}}, \Theta_0]$ and q_{i-1} if $\Theta \in A_i^+ \equiv [\Theta_0, \Theta_0^{q_i}]$. As a consequence in we consider the return times in A_i the relative measures of points that come back to A_i- and to A_i^+ will be given by the Lebesgue

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measure of these intervals normalized to the Lebesgue measure of A_i. As a consequence the weights w_-, w_+ for the return times q_i and q_{i-1} are given by

$$w_- = \frac{\mu_L(A_i^-)}{\mu_L(A_i)} = \frac{1}{1+r} \qquad w_+ = \frac{\mu_L(A_i^+)}{\mu_L(A_i)} = \frac{1}{1+r^{-1}} \qquad r = \frac{\mu_L(A_i^+)}{\mu_L(A_i^-)} \quad (24)$$

We notice that these weights for a quadratic irrational are independent of i. Indeed in this case $p_i = q_{i-1}$ and letting $\omega = [a, a, a, a, \ldots]$

$$r = \frac{\omega\, q_i - q_{i-1}}{q_{i-2} - \omega\, q_{i-1}} = \frac{1}{a + \omega} \tag{25}$$

Indeed taking into account that $\omega(a + \omega) = 1$, the above relation is verified if $q_i - (a + \omega)q_{i-1} - qi - 2 + \omega qi - 1 = q_i - aq_{i-1} - q_{i-2}$, which is the case due to the recurrence for q_i. As a consequence the average time is

$$\langle \tau_{A_i} \rangle = w_-\, q_i + w_+\, q_{i-i} = \frac{(a + \omega)\, q_i + q_{i-1}}{1 + a + \omega} \tag{26}$$

The normalized times in the limit $i \to \infty$ become

$$t_- = \lim_{i\to\infty} \frac{q_i}{\langle \tau_{A_i} \rangle} = (w_+ + \omega w_+)^{-1} = \frac{1 + a + \omega}{a + 2\omega}$$

$$t_+ = \lim_{i\to\infty} \frac{q_{i-1}}{\langle \tau_{A_i} \rangle} = (w_- + \omega w_+)^{-1} = \frac{\omega\,(1 + a + \omega)}{a + 2\omega} = \omega\, t_- \tag{27}$$

As a consequence the limit point spectrum exists and is given by

$$F(t) = w_-\, \delta(t - t_-) + w_+\, \delta(t - t_+) \tag{28}$$

We believe that the spectrum exists for any irrational ω. The existence of a spectrum for a generic nested sequence of intervals, squeezing to θ_0, remains an open question.

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FIGURE CAPTIONS

Figure 1a,b,c,d: Illustrate the sub-critical and critical cases with $\omega = \gamma$:(a)Three return times for $k = 1; \Omega = 0.606661; A = (0, 0.0015)$, and densities of the unique invariant ergodic measure for (b) $k = 0.75, \Omega = 0.61088669$ (c) $k = 0.9, \Omega = 0.6083938$ and (d) $k = 1, \Omega = 0.606661$.

Figure 2a,b,c,d: Illustrate the distribution of the return times into a sequence of sub-intervals for the fixed weakly super-critical map $k = 1.01; \Omega = 0.606494989$.

Figures 3a,b,c,d: Illustrate the return times into the interval $(0, 0.0015)$ for weakly super-critical maps with $(k, \Omega) = $ (a) $(1.015, 0.6063931)$ (b) $(1.1, 0.6054765)$ (c) $(1.2, 0.6047099)$ (d) shows $\ln F_\theta(t)$.vs.t for $k = 1.2, \Omega = 0.6047099$, with the slope 0.9996.

Figures 4a,b: Strongly super-critical dynamics: (a) $\ln F_\theta(t)$.vs.t with slope ≈ 1.0 for $(k, \Omega) = (1.3, 0.606187), (3, 0.51738887), (4, 0.36176849), (5, 0.20197433)$, $(6.14, 0.7184975)$ and for $(k, \Omega) = (4.605, 0), (2\Pi, 0), (1.98, 1/2), (2, 1/2)$;

b)$\ln F_A(t)$.vs.t with slope $\neq 1$, for $(k, \Omega) = (4.60366, 0)$ when $A = (0.01)$; $(0, 0.0005)$ and $A = (0, 0.0001)$, with approximation (14) (dash-ed line). Other curves represent $\ln F_A\theta(t)$.vs.t at unstable periodic points for other examples with $\Omega = 0; 1/2$. The maximal slope is 0.85.

a) k=1

b) k=0.75

c) k=0.9

d) k=1

a) k=1.01; Δ=0.0015

b) k=1.01; Δ=0.001

c) k=1.01; Δ=0.0001

d) Δ=0.0001; k=1.01
slope=1.02

k=1.015 ;Δ=0.0015

k=1.1 Δ=0.0015

c)k=1.2;Δ=0.0015

d) k=1.2; Δ=0.0015
slope=0.9996

a) b)